## PHYSICS AND MATHS

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# WEAKLY NONLINEAR INTEGRODIFFERENTIAL EQUATION

The conditions for the existence of solutions of boundary-value problems for weakly nonlinear integrodifferential equations with parameters were given in [1].

The present work considers the questions for the illustration of the main points of theoretical conclusions to the next problem

$$x''(t)+\frac{12}{\pi^2}x(t)=6\pi^2t-12t^3+\lambda\cos\frac{t}{2}+\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\sin(t+s)|x(s)|ds, \qquad (1)$$

$$x(-\pi)=x(\pi)=0, \qquad \int_{-\pi}^{\pi}x(t)\cos\frac{t}{2}dt=0. \qquad (2)$$

We have in (1): $p=\frac{12}{\pi^2}$, $\varepsilon=\frac{1}{2\pi^2}$, $f(t)=6\pi^2t-12t^3$.

We reduce problem (1), (2) to the equivalent integral equation. For this purpose, we consider the auxiliary problem

$$x''(t)=\lambda\cos\frac{t}{2}+y(t), \qquad x(-\pi)=x(\pi)=0, \qquad \int_{-\pi}^{\pi}x(t)\cos\frac{t}{2}dt=0 \qquad (3)$$

As will be seen below, problem (3) has the unique solution for an arbitrary given function $y \in L_2[-\pi, \pi]$. Let us construct it. For this purpose, we will find the general solution of (3). It takes the form

$$x(t) = ct + d - 4\lambda \cos\frac{t}{2} + \int_{-\pi}^{\pi} (t-s)y(s)ds. \quad (4)$$

To determine the unknown parameters $\{c, d, \lambda\} \subset R$, we substitute the general solution (4) in conditions (2). After some appropriate calculations, we obtain

$$d - \pi c = 0, \qquad d + \pi c = \int_{-\pi}^{\pi} (\pi - s)y(s)ds, \qquad 4d - 4\pi\lambda = \int_{-\pi}^{\pi} (4\cos\frac{s}{2} + 2s - 2\pi)y(s)ds$$

Having solved this system of equations, we have

$$d = \frac{1}{2}\int_{-\pi}^{\pi} (\pi - s)y(s)ds, \qquad c = \frac{d}{\pi}, \qquad \lambda = -\frac{1}{2}\int_{-\pi}^{\pi} y(s)\cos\frac{s}{2}ds.$$

Substituting this solution in formula (4), we obtain

$$x(t) = \frac{t+\pi}{2\pi}\int_{-\pi}^{\pi} (\pi - s)y(s)ds - \frac{4}{\pi}\cos\frac{t}{2}\int_{-\pi}^{\pi} y(s)\cos\frac{s}{2}ds + \int_{-\pi}^{\pi} (t-s)y(s)ds.$$

Let us introduce the notation

$$\Gamma(s) = -\frac{1}{\pi}\cos\frac{s}{2}, \qquad G(t,s) = -\frac{4}{\pi}\cos\frac{t}{2}\cos\frac{s}{2} + \frac{1}{2\pi}\begin{cases}(t+\pi)(s-\pi), & t \le s,\\ (t-\pi)(s+\pi), & t \ge s.\end{cases} \quad (5)$$

Then the solution of problem (3) takes the form

$$x(t) = \int_{-\pi}^{\pi} G(t,s)y(s)ds\,, \qquad \lambda = \int_{-\pi}^{\pi} \Gamma(s)y(s)ds\,, \tag{6}$$

i.e., the form (10), (11) in [1], where $h(t)=0,\ \sigma=0$ and the kernels $G(t,s)$ and $\Gamma(s)$ are defined by formula (5).

Obviously, problem (1), (2) will be equivalent to problem (3), if we set

$$y(t) = f(t) - px(t) + \varepsilon \int_{-\pi}^{\pi} \sin(t+s)|x(s)|ds$$

in the latter. Substituting the first relation in (6) in the above formula, we obtain the integral equation

$$y(t) = f(t) - p\int_{-\pi}^{\pi} G(t,s)y(s)ds + \varepsilon \int_{-\pi}^{\pi} \sin(t+\xi)\left|\int_{-\pi}^{\pi}(\xi,s)y(s)ds\right|d\xi\,. \tag{7}$$

By Theorem 1 in [1], problem (1), (2) is equivalent to the integral equation (7), and their solutions are connected by relations (10) and (11) in [1].
Executing the relevant calculations with regard for the formulas

$$\int_{-\pi}^{\pi} \sin s\left|a\sin s + b(\pi^2 s - s^3)\right|ds = 0\,, \qquad \int_{-\pi}^{\pi} \cos s\left|a\sin s + b(\pi^2 s - s^3)\right|ds = (24-2\pi^2)b$$

which are proper for any $a$ and $b$ with $R^+$, we verify that

$$x^*(t) = \pi^2(\sin t + \pi^2 t - t^3)\,, \qquad \lambda^* = 0 \tag{8}$$

is a solution of the problem

$$x''(t) + \frac{12}{\pi^2}x(t) = 6\pi^2 t - 12t^3 + \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \sin(t+s)|x(s)|ds + \lambda\cos\frac{t}{2}\,,$$

$$x(-\pi) = x(\pi) = 0, \qquad \int_{-\pi}^{\pi} x(t)\cos\frac{t}{2}\,dt = 0.$$

Since the calculations with the use of formula (5) yield

$$\int_{-\pi}^{\pi} G(t,s)(-6\pi^2 s - \pi^2 \sin s)ds = \pi^2(\sin t + \pi^2 t - t^3), \tag{9}$$

it is easy to see that the function

$$y^*(t) = -6\pi^2 t - \pi^2 \sin t \tag{10}$$

is a solution of the integral equation (7).

On the basis of formulas (8), (9), and (10), we have

$$x^*(t) = \int_{-\pi}^{\pi} G(t,s)\, y^*(s)ds, \qquad y^*(t) = \frac{d^2}{dt^2} x^*(t)$$

which confirms the assertion of Theorem 1 in [1].

**References:**

1. Luchka, A. Y., & Nesterenko, O. B. (2009). Methods for the solution of boundary-value problems for weakly nonlinear integro-differential equations with parameters and restrictions. *Ukrainian Mathematical Journal*, *61*(5), 801–809. https://doi.org/10.1007/s11253-009-0241-x